1. Define Fourier Transform and its inverse.

Let f(x) be a continuous function of a real variable x. The Fourier transform of f(x) is defined by the equation

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]\,dx$$

Where j = √-1

Given F(u), f(x) can be obtained by using the inverse Fourier transform

$$\Im^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]\,du.$$

The Fourier transform exists if f(x) is continuous and integrable and F(u) is integrable.

The Fourier transform of a real function, is generally complex,

$$F(u) = R(u) + jI(u)$$

Where R(u) and I(u) are the real and imiginary components of F(u). F(u) can be expressed in exponential form as

$$F(u) = |F(u)|\, e^{j\text{Ø}(u)}$$

where

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

and

$$\text{Ø}(u, v) = \tan^{-1}[\, I(u, v)/R(u, v)\,]$$

The magnitude function |F (u)| is called the Fourier Spectrum of f(x) and Φ(u) its phase angle.

The variable u appearing in the Fourier transform is called the frequency variable.

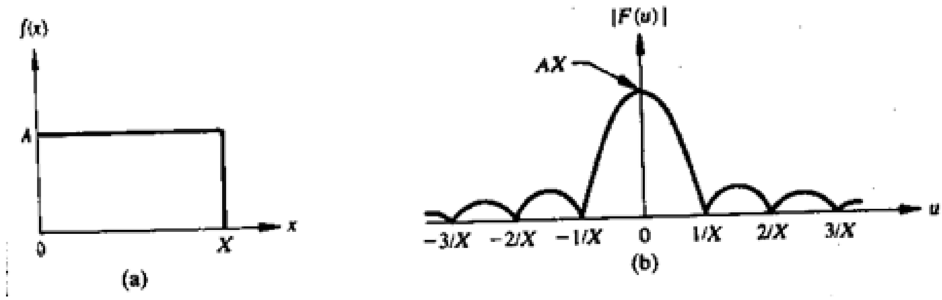

Fig 1 A simple function and its Fourier spectrum

The Fourier transform can be easily extended to a function f(x, y) of two variables**.** If f(x, y) is continuous and integrable and F(u,v) is integrable, following Fourier transform pair exists

$$\Im\{f(x, y)\} = F(u, v) = \iint_{-\infty}^{\infty} f(x, y) \exp[-j2\pi(ux + vy)]\, dx\, dy$$

and

$$\Im^{-1}\{F(u, v)\} = f(x, y) = \iint_{-\infty}^{\infty} F(u, v)\exp[j2\pi(ux + vy)]\, du\, dv$$

\

Where u, v are the frequency variables

The Fourier spectrum, phase, are

$$|F(u, v)| = [R^2(u, v) + I^2(u, v)]^{1/2}$$

$$Ø(u, v) = \tan^{-1}[I(u, v)/R(u, v)]$$

2. Define discrete Fourier transform and its inverse.

The discrete Fourier transform pair that applies to sampled function is given by,

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] \qquad (1)$$

For u = 0, 1, 2 . . . . , N-1, and

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp[j2\pi ux/N] \qquad (2)$$

For x = 0, 1, 2 . . . ., N-1.

In the two variable case the discrete Fourier transform pair is

$$F(u, v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x, y)\exp[-j2\pi(ux/M + vy/N)]$$

For u = 0, 1, 2 . . . , M-1, v = 0, 1, 2 . . . , N - 1, and

$$f(x, y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u, v)\exp[j2\pi(ux/M + vy/N)]$$

For x = 0, 1, 2 . . . , M-1, y = 0, 1, 2 . . . , N-1.

If M = N, then discrete Fourier transform pair is

$$F(u, v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x, y)\exp[-j2\pi(ux + vy)/N]$$

For u, v = 0, 1, 2 . . . , N – 1, and

$$f(x, y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u, v)\exp[j2\pi(ux + vy)/N]$$

For x, y = 0, 1, 2 . . . , N – 1

3. State and prove separability property of 2D-DFT.

The separability property of 2D-DFT states that, the discrete Fourier transform pair can be expressed in the separable forms. i.e. ,

$$F(u, v) = \frac{1}{N}\sum_{x=0}^{N-1} \exp[-j2\pi ux/N] \sum_{y=0}^{N-1} f(x, y)\exp[-j2\pi vy/N] \quad (1)$$

For u, v = 0, 1, 2 . . . , N – 1, and

$$f(x, y) = \frac{1}{N}\sum_{u=0}^{N-1} \exp[j2\pi ux/N] \sum_{v=0}^{N-1} F(u, v)\exp[j2\pi vy/N] \quad (2)$$

For x, y = 0, 1, 2 . . . , N – 1

The principal advantage of the separability property is that F(u,v) or f(x,y) can be obtained in two steps by successive applications of the 1-D Fourier transform or its inverse. This advantage becomes evident if equation (1) is expressed in the form

$$F(u, v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x, v)\exp[-j2\pi ux/N] \quad (3)$$

Where,

$$F(x, v) = N\left[\frac{1}{N}\sum_{y=0}^{N-1} f(x, y)\exp[-j2\pi vy/N]\right]. \quad (4)$$

For each value of x, the expression inside the brackets in eq(4) is a 1-D transform, with frequency values v = 0, 1, . . . , N-1. Therefore the 2-D function f(x, v) is obtained by taking a transform along each row of f(x, y) and multiplying the result by N. The desired result, F(u, v), is then obtained by taking a transform along each column of F(x, v), as indicated by eq(3)

4. State and prove the translation property.

The translation properties of the Fourier transform pair are

$$f(x, y)\exp[j2\pi(u_0x + v_0y)/N] \Leftrightarrow F(u - u_0, v - v_0) \quad (1)$$

and

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)\exp[-j2\pi(ux_0 + vy_0)/N] \quad (2)$$

Where the double arrow indicates the correspondence between a function and its Fourier Transform,

Equation (1) shows that multiplying f(x, y) by the indicated exponential term and taking the transform of the product results in a shift of the origin of the frequency plane to the point (u_o, v_o).

Consider the equation (1) with $u_o = v_o = N/2$ or

$$\exp[j2\Pi(u_o x + v_o y)/N] = e^{j\Pi(x + y)}$$

$$= (-1)^{(x + y)}$$

and

$$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - N/2, v - N/2)$$

Thus the origin of the Fourier transform of f(x, y) can be moved to the center of its corresponding N x N frequency square simply by multiplying f(x, y) by $(-1)^{x+y}$. In the one variable case this shift reduces to multiplication of f(x) by the term $(-1)^x$. Note from equation (2) that a shift in f(x, y) does not affect the magnitude of its Fourier transform as,

$$|F(u, v)\exp[-j2\pi(ux_0 + vy_0)/N]| = |F(u, v)|.$$

4. State distributivity and scaling property.

Distributivity:

From the definition of the continuous or discrete transform pair,

$$\Im\{f_1(x, y) + f_2(x, y)\} = \Im\{f_1(x, y)\} + \Im\{f_2(x, y)\}$$

and, in general,

$$\Im\{f_1(x, y) \cdot f_2(x, y)\} \neq \Im\{f_1(x, y)\} \cdot \Im\{f_2(x, y)\}.$$

In other words, the Fourier transform and its inverse are distributive over addition but not over multiplication.

Scaling:

For two scalars a and b,

$$af(x, y) \Leftrightarrow aF(u, v)$$

$$f(ax, by) \Leftrightarrow \frac{1}{|ab|} F(u/a, v/b).$$

5. Explain the basic principle of Hotelling transform.

Hotelling transform:

The basic principle of hotelling transform is the statistical properties of vector representation. Consider a population of random vectors of the form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

And the mean vector of the population is defined as the expected value of x i.e.,

$$m_x = E\{x\}$$

The suffix m represents that the mean is associated with the population of x vectors. The expected value of a vector or matrix is obtained by taking the expected value of each elememt. The covariance matrix C_x in terms of x and m_x is given as

$$C_x = E\{(x-m_x)(x-m_x)^T\}$$

T denotes the transpose operation. Since, x is n dimensional, $\{(x-m_x)(x-m_x)^T\}$ will be of n x n dimension. The covariance matrix is real and symmetric. If elements x_i and x_j are uncorrelated, their covariance is zero and, therefore, $c_{ij} = c_{ji} = 0$.

For M vector samples from a random population, the mean vector and covariance matrix can be approximated from the samples by

$$\mathbf{m}_x = \frac{1}{M}\sum_{k=1}^{M}\mathbf{x}_k$$

and

$$\mathbf{C}_x = \frac{1}{M}\sum_{k=1}^{M}\mathbf{x}_k\mathbf{x}_k^T - \mathbf{m}_x\mathbf{m}_x^T.$$

6. Write about Slant transform.

The Slant transform matrix of order N x N is the recursive expression S_n is given by

$$\mathbf{S}_N = \frac{1}{\sqrt{2}}\left[\begin{array}{cc|c|cc|c} 1 & 0 & \mathbf{0} & 1 & 0 & \mathbf{0} \\ a_N & b_N & & -a_N & b_N & \\ \hline \mathbf{0} & & \mathbf{I}_{(N/2)-2} & \mathbf{0} & & \mathbf{I}_{(N/2)-2} \\ \hline 0 & 1 & \mathbf{0} & 0 & -1 & \mathbf{0} \\ -b_N & a_N & & b_N & a_N & \\ \hline \mathbf{0} & & \mathbf{I}_{(N/2)-2} & \mathbf{0} & & -\mathbf{I}_{(N/2)-2} \end{array}\right]\left[\begin{array}{c|c} \mathbf{S}_{N/2} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{S}_{N/2} \end{array}\right]$$

Where I_m is the identity matrix of order M x M, and

$$\mathbf{S}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The coefficients are

$$a_N = \left[\frac{3N^2}{4(N^2-1)}\right]^{1/2}$$

and

$$b_N = \left[\frac{N^2 - 4}{4(N^2 - 1)}\right]^{1/2}$$

The slant transform for N = 4 will be

$$S_4 = \frac{1}{\sqrt{4}}\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{3}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \\ \frac{1}{\sqrt{5}} & \frac{-3}{\sqrt{5}} & \frac{3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \end{bmatrix}$$

7. What are the properties of Slant transform?

Properties of Slant transform

(i) The slant transform is real and orthogonal.

$$S = S^*$$

$$S^{-1} = S^T$$

(ii) The slant transform is fast, it can be implemented in ($N \log_2 N$) operations on an N x 1 vector.

(iii) The energy deal for images in this transform is rated in very good to excellent range.

(iv) The mean vectors for slant transform matrix S are not sequentially ordered for $n \geq 3$.

8. Define discrete cosine transform.

The 1-D discrete cosine transform is defined as

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{(2x+1)u\pi}{2N}\right]$$

For u = 0, 1, 2, . . , N-1. Similarly the inverse DCT is defined as

$$f(x) = \sum_{u=0}^{N-1} \alpha(u)C(u)\cos\left[\frac{(2x+1)u\pi}{2N}\right]$$

For u = 0, 1, 2, . . , N-1

Where α is

$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & \text{for } u = 0 \\ \sqrt{\frac{2}{N}} & \text{for } u = 1, 2, \ldots, N-1. \end{cases}$$

The corresponding 2-D DCT pair is

$$C(u, v) = \alpha(u)\alpha(v) \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y)\cos\left[\frac{(2x+1)u\pi}{2N}\right]\cos\left[\frac{(2y+1)v\pi}{2N}\right]$$

For u, v = 0, 1, 2, . . , N-1, and

$$f(x, y) = \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \alpha(u)\alpha(v)C(u, v)\cos\left[\frac{(2x+1)u\pi}{2N}\right]\cos\left[\frac{(2y+1)v\pi}{2N}\right]$$

For x, y= 0, 1, 2, . . , N-1

9. Explain about Haar transform.

The Haar transform is based on the Haar functions, $h_k(z)$, which are defined over the continuous, closed interval z ε [0, 1], and for k = 0, 1, 2 . . . , N-1, where $N = 2^n$. The first step in generating the Haar transform is to note that the integer k can be decomposed uniquely as

$$k = 2^p + q - 1$$

where $0 \leq p \leq n-1$, q = 0 or 1 for p = 0, and $1 \leq q \leq 2^p$ for $p \neq 0$. For example, if N = 4, k, q, p have following values

k	p	q
0	0	0
1	0	1
2	1	1
3	1	2

The Haar functions are defined as

$$h_0(z) \triangleq h_{00}(z) = \frac{1}{\sqrt{N}} \text{ for } z \in [0, 1] \ldots\ldots (1)$$

and

$$h_k(z) \triangleq h_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & \frac{q-1}{2^p} \le z < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \le z < \frac{q}{2^p} \\ 0 & \text{otherwise for } z \in [0, 1]. \end{cases}$$

These results allow derivation of Haar transformation matrices of order N x N by formation of the *i*th row of a Haar matrix from elements oh $h_i(z)$ for z = 0/N, 1/N, . . . , (N-1)/N. For instance, when N = 2, the first row of the 2 x 2 Haar matrix is computed by using $h_o(z)$ with z = 0/2, 1/2. From equation (1) , $h_o(z)$ is equal to $1/\sqrt{2}$, independent of z, so the first row of the matrix has two identical $1/\sqrt{2}$ elements. Similarly row is computed. The 2 x 2 Haar matrix is

$$\mathbf{A}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Similarly matrix for N = 4 is

$$\mathbf{A}_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

10. What are the properties of Haar transform.

Properties of Haar transform:

1. The Haar transform is real and orthogonal.

2. The Haar transform is very fast. It can implement O(n) operations on an N x 1 vector.

3. The mean vectors of the Haar matrix are sequentially ordered.

4. It has a poor energy deal for images.

11. Write a short notes on Hadamard transform.

1-D forward kernel for hadamard transform is

$$g(x, u) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1} b_i(x)b_i(u)}$$

Expression for the 1-D forward Hadamard transform is

$$H(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)(-1)^{\sum_{i=0}^{n-1} b_i(x)b_i(u)}$$

Where $N = 2^n$ and u has values in the range 0, 1, . . . , N-1.

1-D inverse kernel for hadamard transform is

$$h(x, u) = (-1)^{\sum_{i=0}^{n-1} b_i(x)b_i(u)}$$

Expression for the 1-D inverse Hadamard transform is

$$f(x) = \sum_{u=0}^{N-1} H(u)(-1)^{\sum_{i=0}^{n-1} b_i(x)b_i(u)}$$

The 2-D kernels are given by the relations

$$g(x, y, u, v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u) + b_i(y)b_i(v)]}$$

and

$$h(x, y, u, v) = \frac{1}{N}(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u) + b_i(y)b_i(v)]}$$

2-D Hadamard transform pair is given by following equations

$$H(u, v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x, y)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u) + b_i(y)b_i(v)]}$$

$$f(x, y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} H(u, v)(-1)^{\sum_{i=0}^{n-1}[b_i(x)b_i(u) + b_i(y)b_i(v)]}$$

Values of the 1-D hadamard transform kernel for N = 8 is

u \ x	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	−	+	−	+	−	+	−
2	+	+	−	−	+	+	−	−
3	+	−	−	+	+	−	−	+
4	+	+	+	+	−	−	−	−
5	+	−	+	−	−	+	−	+
6	+	+	−	−	−	−	+	+
7	+	−	−	+	−	+	+	−

The Hadamard matrix of lowest order N = 2 is

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If $\mathbf{H}_N$ represents the matrix of order N, the recursive relationship is given by

$$\mathbf{H}_{2N} = \begin{bmatrix} \mathbf{H}_N & \mathbf{H}_N \\ \mathbf{H}_N & -\mathbf{H}_N \end{bmatrix}$$

Where $\mathbf{H}_{2N}$ is the Hadamard matrix of order 2N and $N = 2^n$

12. Write about Walsh transform.

The discrete Walsh transform of a function *f* (x), denoted *W(u),* is given by

$$W(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u)}$$

Walsh transform kernel is symmetric matrix having orthogonal rows and columns. These properties, which hold in general, lead to an inverse kernel given by

$$h(x, u) = \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u)}.$$

Thus the inverse Walsh transform is given by

$$f(x) = \sum_{u=0}^{N-1} W(u) \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u)}.$$

The 2-D forward and inverse Walsh kernels are given by

$$g(x, y, u, v) = \frac{1}{N}\prod_{i=0}^{n-1} (-1)^{[b_i(x)b_{n-1-i}(u) + b_i(y)b_{n-1-i}(v)]}$$

and

$$h(x, y, u, v) = \frac{1}{N}\prod_{i=0}^{n-1} (-1)^{[b_i(x)b_{n-1-i}(u) + b_i(y)b_{n-1-i}(v)]}.$$

Thus the forward and inverse Walsh transforms for 2-D are given by

$$W(u, v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x, y) \prod_{i=0}^{n-1} (-1)^{[b_i(x)b_{n-1-i}(u) + b_i(y)b_{n-1-i}(v)]}$$

and

$$f(x, y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} W(u, v) \prod_{i=0}^{n-1} (-1)^{[b_i(x)b_{n-1-i}(u) + b_i(y)b_{n-1-i}(v)]}.$$

The Walsh Transform kernels are seperable and symmetric, because

$$\begin{aligned} g(x, y, u, v) &= g_1(x, u)g_1(y, v) \\ &= h_1(x, u)h_1(y, v) \\ &= \left[\frac{1}{\sqrt{N}}\prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-1-i}(u)}\right]\left[\frac{1}{\sqrt{N}}\prod_{i=0}^{n-1} (-1)^{b_i(y)b_{n-1-i}(v)}\right]. \end{aligned}$$

Values of the 1-D walsh transform kernel for N = 8 is

u \ x	0	1	2	3	4	5	6	7
0	+	+	+	+	+	+	+	+
1	+	+	+	+	−	−	−	−
2	+	+	−	−	+	+	−	−
3	+	+	−	−	−	−	+	+
4	+	−	+	−	+	−	+	−
5	+	−	+	−	−	+	−	+
6	+	−	−	+	+	−	−	+
7	+	−	−	+	−	+	+	−